AP CALCULUS BC

Definition of e: $e = \lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n$

Absolute value: $|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$

End Behavior of a Function: $\lim_{x \to \pm\infty} f(x)$

Limit Rules to Know:

$\lim_{x \to a} f(x) = \frac{0}{0}$ - factor, use the conjugate method, or clean up messy fractions

$\lim_{x \to a} f(x) = \frac{\#}{0}$ - vertical asymptote - only 3 possible answers $\infty, -\infty,$ or DNE

$\lim_{x \to \infty} = \frac{\text{slow}}{\text{fast}} = 0$

$\lim_{x \to \infty} = \frac{\text{fast}}{\text{slow}} = \pm\infty$ - follow rules for top function (Growth order: e→p→l)

Trig limits:

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

$\lim_{x \to 0} \frac{\sin ax}{x} = a$

$\lim_{x \to 0} \frac{\sin ax}{bx} = \frac{a}{b}$

$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

Definition of continuity: f is continuous at x = c if

1) $f(c)$ is defined;
2) $\lim_{x \to c} f(x)$ exists;
3) $\lim_{x \to c} f(x) = f(c)$.

Definition of the derivative:

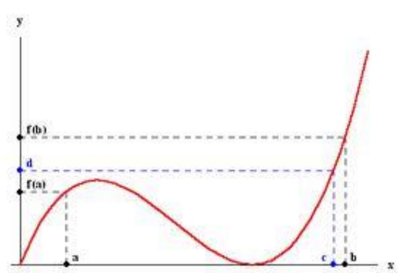
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$ (Alternative form)

Average rate of change (slope of the secant line): of $f(x)$ on [a, b] = $\frac{f(b) - f(a)}{b - a}$

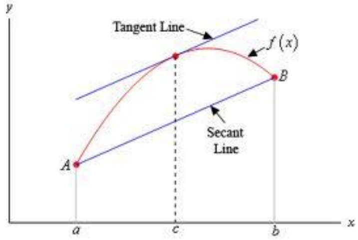
Intermediate Value Theorem

If f is continuous on $[a, b]$ and d is any number between $f(a)$ and $f(b)$, then there is at least one number c between a and b such that $f(c) = d$.



Mean Value Theorem

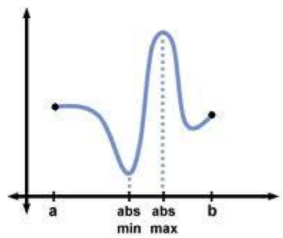
If f(x) is continuous [a,b] and f(x) is differentiable (a,b) then there exists at least one c on (a,b) such that



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Extreme Value Theorem

If a function is continuous on a closed interval, then the function is guaranteed to have an absolute maximum and an absolute minimum in the interval.



Derivative Rules:

$$\frac{d}{dx}c = 0$$

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}f(x)g(x) = f(x)g'(x) + g(x)f'(x)$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\csc x = -\csc x \cot x$$

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}a^x = a^x \ln a$$

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

$$\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$$

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\cot^{-1} x = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\csc^{-1} x = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{dy}{dx}f^{-1}(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

Definition of a Critical Number:

Let f be defined at c. If $f'(c)=0$ or if f' is undefined at c, then c is a critical **number** of f.

$f(c)$ is a critical **value**

First Derivative Test:

Let c be a critical number of a function f that is continuous on an open interval I containing c. If f is differentiable on the interval, except possibly at c, then $f(c)$ can be classified as follows.

1) If $f'(x)$ changes from negative to positive at c, then $f(c)$ is a **relative minimum** of f.

2) If $f'(x)$ changes from positive to negative at c, then $f(c)$ is a **relative maximum** of f.

Second Derivative Test for determining maximums and minimums:

Let f be a function such that the second derivative of f exists on an open interval containing c.

1) If $f'(c)=0$ and $f''(c)>0$, then $f(c)$ is a relative minimum.

2) If $f'(c)=0$ and $f''(c)<0$, then $f(c)$ is a relative maximum.

Definition of Concavity:

Let f be differentiable on an open interval I. The graph of f is **concave upward** on I if f' is increasing on the interval and **concave downward** on I if f' is decreasing on the interval.

Test for Concavity:

Let f be a function whose second derivative exists on an open interval I.

1) If $f''(x)>0$ for all x in I, then the graph of f is concave upward in I.

2) If $f''(x)<0$ for all x in I, then the graph of f is concave downward in I.

Definition of an Inflection Point:

A function f has an inflection point at $(c, f(c))$

1) if $f''(c)=0$ or $f''(c)$ does not exist <u>and</u>

2) if f'' changes sign from positive to negative or negative to positive at $x=c$

<u>OR</u> if $f'(x)$ changes from increasing to decreasing or decreasing to increasing at $x=c$.

Estimating the Area under the Curve

$\Delta x=\frac{b-a}{n}$

Riemann Sums:

LRAM: $A=\frac{b-a}{n}(\textit{sum of } n \textit{ heights from the left side})$

RRAM: $A=\frac{b-a}{n}(\textit{sum of } n \textit{ heights from the right side})$

MRAM: $A=\frac{b-a}{n}(\textit{sum of } n \textit{ heights from the middle})$

Trapezoidal Rule:

$A=\frac{1}{2}\cdot\frac{b-a}{n}\left[1^{st} \textit{ height}+2(\textit{middle heights})+\textit{last height}\right]$

* If x-values are NOT evenly divided the formulas for Riemann Sums and Trapezoidal Rule do NOT apply!

Integral Rules:

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$

$\int \frac{1}{x}\,dx = \ln x + c$

$\int e^x\,dx = e^x + c$

$\int \sin x\,dx = -\cos x + c$

$\int \cos x\,dx = \sin x + c$

$\int \sec^2 x\,dx = \tan x + c$

$\int \csc^2 x\,dx = -\cot x + c$

$\int \sec x \cdot \tan x\,dx = \sec x + c$

$\int \csc x \cdot \cot x\,dx = -\csc x + c$

$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$

$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$

$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1} x + c$

U-Substitution Shortcuts:

$\int \sin kx\,dx = \frac{-\cos kx}{k} + c$

$\int \cos kx\,dx = \frac{\sin kx}{k} + c$

$\int \sec^2 kx\,dx = \frac{\tan kx}{k} + c$

$\int \csc^2 kx\,dx = \frac{-\cot kx}{k} + c$

$\int \sec kx \cdot \tan kx\,dx = \frac{\sec kx}{k} + c$

$\int \csc kx \cdot \cot kx\,dx = \frac{-\csc kx}{k} + c$

$\int e^{kx}\,dx = \frac{e^{kx}}{k} + c$

Fundamental Theorem of Calculus

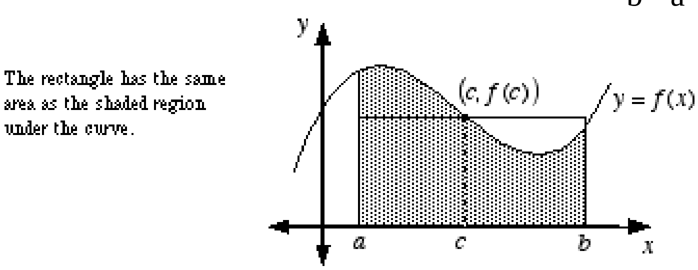
First Fundamental Theorem of Calculus: $\int_a^b f'(x)\,dx = f(b) - f(a)$

Second Fundamental Theorem of Calculus: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

Average value of $f(x)$ on $[a, b]$: $f_{AVE} = \frac{1}{b-a}\int_a^b f(x)\,dx$

Mean Value Theorem (Integrals)

If f(x) is continuous [a,b], then there exists a value c in [a,b] such that $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$.



Slope Fields: used to solve differential equations

* small segments satisfy dy/dx
* the image you see is the solution to the differential equation (y =)

Euler's Method: used to estimate "y"

(x, y) (given)	$\frac{dy}{dx}$ (given)	Δx (same throughout)	$\Delta y\left(= \frac{dy}{dx}\Delta x\right)$	$(x+\Delta x, y+\Delta y)$

Integration by Parts:

Indefinite: $\int u\,dv = uv - \int v\,du$ Definite: $\int_a^b u\,dv = \left[uv - \int v\,du \right]_a^b$

<u>Order for choosing "u"</u>

L – Logarithms
I – Inverse Trig Functions
P – Polynomials
E – Exponential Functions
T – Trig Functions

Exponential Growth and Decay

<u>Law of exponential change</u> – $\frac{dy}{dx} = ky$; y is a rate proportional to the amount present

<u>Exponential Growth and Decay</u> - $y = Ce^{kt}$

Newton's Law of Cooling

<u>Differential Equation</u> <u>Newton's Law of Cooling Formula</u>

$\frac{dT}{dt} = -k\left(T - T_s\right) \quad \rightarrow \quad T - T_s = \left(T_o - T_s\right)e^{-kt}$

Logistic Growth

<u>Logistic Differential equation</u>

$\frac{dP}{dt} = KP\left(A - P\right)$ } integration (by partial fraction decomposition)

A is the carrying capacity $\Rightarrow \lim_{t \to \infty} P(t) = A$

P is growing the fastest at half the carrying capacity $\Rightarrow \frac{dP}{dt}$ is a max when $P = \frac{1}{2}A$

<u>Logistic Growth Formula</u>

$$P = \frac{A}{1 + Ce^{-Ak\,t}}$$

Area Under a Curve

$A = \int_a^b \left[f(x) - g(x) \right] dx$ or $A = \int_a^b \left[upper - lower \right] dx$

*Solve equations for y
*limits of integration come from the x-axis

$A = \int_c^d \left[f(y) - g(y) \right] dy$ or $A = \int_c^d \left[right - left \right] dy$

*Solve equations for x
*limits of integration come from the y-axis

L'Hospital's Rule:

Used to evaluate limits that produce the following indeterminate forms: $\frac{0}{0}, \frac{\infty}{\infty}, \underbrace{\infty - \infty,\ 0 \cdot \infty,\ 0^0, 1^\infty, \infty^0}_{\text{transform to } \frac{0}{0} \text{ or } \frac{\infty}{\infty}}$

If $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ *or* $\frac{\pm\infty}{\pm\infty}$, *then* $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, *where a is finite, finite*$^{\pm}$, *or* $\pm\infty$

Volume

Disk Method

A. Horizontal Axis of Rotation: $V = \pi \int_a^b (upper - lower)^2 dx$

*Representative rectangle is perpendicular to horizontal axis of rotation
*Solve equations for y
*limits of integration come from the x-axis

B. Vertical Axis of Rotation: $V = \pi \int_c^d (right - left)^2 \, dy$

*Representative rectangle is perpendicular to vertical axis of rotation
*Solve equations for x
*limits of integration come from the y-axis

Washer Method

A. Horizontal Axis of Rotation: $V = \pi \int_a^b \left[\underbrace{(upper - lower)^2}_{whole} - \underbrace{(upper - lower)^2}_{hole} \right] dx$

*Representative rectangle of the "whole" is perpendicular to horizontal axis of rotation
* Representative rectangle of the "hole" is perpendicular to horizontal axis of rotation
*Solve equations for y
*limits of integration come from the x-axis

B. Vertical Axis of Rotation: $V = \pi \int_c^d \left[\underbrace{(right - left)^2}_{whole} - \underbrace{(right - left)^2}_{hole} \right] dy$

*Representative rectangle of the "whole" is perpendicular to vertical axis of rotation
* Representative rectangle of the "hole" is perpendicular to vertical axis of rotation
*Solve equations for x
*limits of integration come from the y-axis

Cross Sections

A. Cross sections are perpendicular to x-axis $V = \int_a^b A(x) \, dx$

*solve equations for y
*representative rectangle is the "side" of the cross section
* length of the representative rectangle is "upper – lower"
*limits of integration come from the x-axis

B. Cross sections are perpendicular to y-axis $V = \int_c^d A(y) \, dy$

*solve equations for x
*representative rectangle is the "side" of the cross section
* length of the representative rectangle is "right – left"
*limits of integration come from the y-axis

Length of a Curve:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx} \right)^2} \, dx$$

*solve equations for y
*limits of integration come from the x-axis

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy} \right)^2} \, dy$$

*solve equations for x
*limits of integration come from the y-axis

Horizontal or Vertical Motion:

If an object moves along a straight line with position function $s(t)$, then its

Velocity is $v(t)=s'(t)$

Speed = $|v(t)|$

Acceleration is $a(t)=v'(t)=s''(t)$

Displacement (change in position) from $x=a$ to $x=b$ is Displacement = $\int_a^b v(t)\,dt$

Position = Initial position + displacement

Total Distance traveled from $x=a$ to $x=b$ is Total Distance = $\int_a^b |v(t)\,dt|$

or (without a calculator) Total Distance = $\left|\int_a^c v(t)\,dt\right|+\left|\int_c^b v(t)\,dt\right|$, where $v(t)$ changes sign at $x=c$.

Parametric:

1st Derivative - $\dfrac{dy}{dx}=\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$ 2nd Derivative - $\dfrac{d^2y}{dx^2}=\dfrac{\frac{dy'}{dt}}{\frac{dx}{dt}}$ Arc Length - $L=\int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

Vector:

Position vector = $\langle x(t),\ y(t)\rangle$

Velocity vector = $\langle x'(t),\ y'(t)\rangle$

Acceleration vector = $\langle x''(t),\ y''(t)\rangle$

Speed (or magnitude of velocity vector) = $|v(t)|=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$

Displacement = $\left\langle \int_{t_1}^{t_2} v_1(t)\,dt, \int_{t_1}^{t_2} v_2(t)\,dt \right\rangle$

Distance traveled from $t=a$ to $t=b$ (or length of arc) is $\int_{t_1}^{t_2}|v(t)|\,dt=\int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

Polar:

$x=r\cos\theta$ and $y=r\sin\theta$

1st derivative - $\dfrac{dy}{dx}=\dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

Area inside a polar curve - $A=\dfrac{1}{2}\int_a^b r^2\,d\theta$

Area between 2 polar curves - $A=\dfrac{1}{2}\int_\alpha^\beta \left(r_o^{\,2}-r_i^{\,2}\right)d\theta$

Summary of Tests for Series

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} ar^n$	$\|r\| < 1$	$\|r\| \geq 1$	Sum: $S = \frac{a}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+1})$	$\lim_{n\to\infty} b_n = L$		Sum: $S = b_1 - L$
p-Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$	
Alternating Series	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$0 < a_{n+1} \leq a_n$ and $\lim_{n\to\infty} a_n = 0$		Remainder: $\|R_N\| \leq a_{N+1}$
Integral (f is continuous, positive, and decreasing)	$\sum_{n=1}^{\infty} a_n$, $a_n = f(n) \geq 0$	$\int_1^{\infty} f(x)\,dx$ converges	$\int_1^{\infty} f(x)\,dx$ diverges	Remainder: $0 < R_N < \int_N^{\infty} f(x)\,dx$
Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\|a_n\|} < 1$	$\lim_{n\to\infty} \sqrt[n]{\|a_n\|} > 1$	Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{\|a_n\|} = 1$.
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| < 1$	$\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| > 1$	Test is inconclusive if $\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| = 1$.
Direct Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges	
Limit Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges	

Definition of a Taylor polynomial:

If f has n derivatives at c, then the polynomial

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the **nth Taylor polynomial for f at c.**

Lagrange Error Bound for a Taylor Polynomial (or Taylor's Theorem Remainder):

If f is differentiable through order $n+1$ in an interval I containing c, then for each x in I, there exists z

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x),$$

between x and c such that where $R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$. $R_n(x)$ gives a bound for the size of the error found by the nth degree Taylor polynomial.

The remainder represents the difference between the function and the polynomial. That is,

$$|R_n| = |f(x) - P_n(x)| .$$

Alternating Series Remainder:

If a series has terms that alternate, decrease in absolute value, and have a limit of 0 (so that the series converges by the Alternating Series Test), then the absolute value of the remainder R_n involved in approximating the sum S by S_n is less than the first neglected term. That is,

$$|R_n| = |S - S_n| < a_{n+1} .$$

Maclaurin series that you must know:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

geometric series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$